On the Hamiltonicity of the Cartesian Product

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Abstract

We examine the hamiltonicity of the cartesian product $P = G_1 \times G_2$ of two graphs G_1, G_2. We provide necessary and/or sufficient conditions for P to be hamiltonian, depending on the hamiltonian properties of G_1 and G_2, with corresponding constructions. We also prove a conjecture by Batagelj and Pisanski related to the 'cyclic hamiltonicity' of a graph.

Keywords: cartesian product, graphs, hamiltonian cycles, interconnection networks

1 Introduction

An undirected graph $G = (V, E)$ is said to be *hamiltonian* if it contains a spanning cycle. If it contains a spanning path then G is called *traceable*. The problem of determining whether a graph is hamiltonian or traceable has been fundamental in graph theory. This and related problems have been extensively surveyed (see for example [3, 6]).

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We are interested in the hamiltonicity of the cartesian product of graphs. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their cartesian product is defined as the graph $P = (V, E) = G_1 \times G_2$, where $V = V_1 \times V_2 = \{(u, v) \,|\, u \in V_1, v \in V_2\}$ and

$$E = \{(u, v)(u', v') \,|\, (u = u' \text{ and } vv' \in E_2) \text{ or } (v = v' \text{ and } uu' \in E_1)\}.$$

The motivation behind this work comes from multicomputer interconnection networks, which are almost exclusively based on cartesian products of graphs (e.g. hypercubes, meshes, tori). In such settings, having a hamiltonian cycle is crucial for achieving optimal performance in various communication operations, gossiping being just one example [4, 5].

The problem we study is the following: given two graphs G_1 and G_2, are there any necessary and/or sufficient conditions for their cartesian product $P = G_1 \times G_2$ to be hamiltonian? Although, general conditions for the hamiltonicity of arbitrary graphs are well known, there are few results related to the cartesian product of two graphs [1, 2, 8, 9, 10, 11]. In this paper we summarize previously known results and provide new ones for the hamiltonicity problem in the cartesian product. We also prove a conjecture by Batagelj and Pisanski [1] related to the 'cyclic hamiltonicity' of a graph and give a linear-time algorithm for constructing a hamiltonian cycle in the product of a hamiltonian and certain non-hamiltonian graphs.

2 Terminology and Notation

The notation and terminology we use follows Harary [7]. Let $G = (V, E)$ be an undirected graph with vertex or node set V and edge set E. The number of vertices $|V|$ is known as the *order* of G. Two nodes $v, u \in V$ are *adjacent* if the edge vu is in E. The number of nodes v is adjacent to is the *degree* of v, denoted by $d(v)$. The maximum degree in the graph is $\Delta(G) = \max_{v \in V}\{d(v)\}$.

A *walk* of length k from v to u consists of a sequence of vertices $v{=}v_0, v_1, v_2, ..., v_k{=}u$, and edges v_iv_{i+1}, where $v_iv_{i+1} \in E$, for all $0 \leq i < k$. A *path* is a walk where all vertices are distinct. In the case where all vertices are distinct, except $v = u$, the sequence is a *cycle* of length k, or a k-cycle. The graphs we consider here are

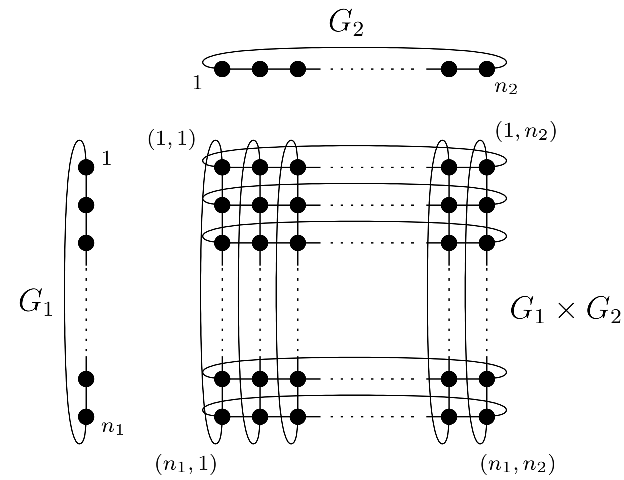


Figure 1: Cartesian product example.

always assumed to be *connected*, i.e. for any pair of vertices there is always a walk between them. If G is connected and has no cycles, then it is called a *tree*. If there exists a path that spans G (i.e. all vertices are included) then G is called *traceable*. If there exists a cycle that spans G then G is called *hamiltonian*.

A graph $G' = (V', E')$ is a *subgraph* of G (denoted by $G' \subseteq G$) if $V' \subseteq V$ and $E' \subseteq E$. If $V' = V$, then G' is a *spanning* subgraph of G. A connected spanning subgraph of G with no cycles is a *spanning tree* of G.

A graph G is *bipartite* if its node set can be partitioned in two disjoint sets $V^{(1)}$ and $V^{(2)}$ such that every edge in G joins a node from $V^{(1)}$ with a node from $V^{(2)}$.

The cartesian product of two graphs G_1 and G_2 was defined in Section 1. The graphs G_1 and G_2 are termed *factors* of the product. An example is given in Figure 1, where the two factors are simple cycles. Clearly, if any of the factors is not connected then their product is also not connected, and hence cannot be hamiltonian; hereafter we assume that all factors are connected.

For a graph G, let us define $\mathcal{D}(G) = \min\{\Delta(T) \,|\, T \text{ is a spanning tree of } G\}$, that is, there is no spanning tree of G with maximum degree less than $\mathcal{D}(G)$.

3 Non-hamiltonian Factors

Consider the case where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are arbitrary graphs. In this section we provide conditions that determine the hamiltonicity of the cartesian product $P = G_1 \times G_2$. As far as we know no other results have appeared in the open literature for this general case.

Theorem 1 *Let G_1 and G_2 be both of odd order. If none contains an odd cycle, then $P = G_1 \times G_2$ is not hamiltonian.*

Proof. A graph has no odd cycles if and only if it is bipartite [7]. It is easy to see that the cartesian product of bipartite graphs is bipartite too [10]: if $V_i^{(1)}$, $V_i^{(2)}$ is the bipartition of G_i, $i = 1, 2$, then a bipartition of P is $V^{(1)}$, $V^{(2)} = V - V^{(1)}$, where $V^{(1)} = \{(v, u) \,|\, (v \in V_1^{(1)} \text{ and } u \in V_2^{(1)}) \text{ or } (v \in V_1^{(2)} \text{ and } u \in V_2^{(2)})\}$. Since bipartite graphs have only even cycles, and the order of P is odd, P cannot contain a hamiltonian cycle. □

The next theorem is based on the maximum degree of the factors' spanning trees.

Theorem 2 *If $\mathcal{D}(G_1) > |V_2| + 1$ or $\mathcal{D}(G_2) > |V_1| + 1$, then $P = G_1 \times G_2$ is not hamiltonian.*

Proof. Without loss of generality, assume that $\mathcal{D}(G_1) > |V_2| + 1$. Suppose for contradiction that the product P is hamiltonian and let $\mathcal{H}$ be a hamiltonian cycle of P. Pick an arbitrary vertex (a, v), mark a as "encountered" and initialize a set A to the empty set. Starting from (a, v), walk along $\mathcal{H}$ and for every two consecutive vertices (b, u) and (b', u') do the following:

> If b' has not been marked yet, mark b' as "encountered" and insert the pair $[b, b']$ in A (note that in this case the definition of the cartesian product implies that $u = u'$ and bb' is an edge of G_1).

Since $\mathcal{H}$ is a hamiltonian cycle, at the completion of the above procedure, all vertices of G_1 are marked as "encountered." Moreover, the fact that a pair $[b, b']$

is inserted in the set A whenever the first vertex (b', u) is visited, where $b' \neq a$ and u is any vertex of G_2, implies that the pairs in A are distinct and their number is $|V_1| - 1$. Additionally, when a pair $[b, b']$ is inserted in A, b' has not been encountered yet; thus, the pairs in A do not induce a cycle. As a consequence, the edges corresponding to the pairs in A form a spanning tree T of G_1.

Consider any vertex b of G_1 and let us compute its degree in T. Clearly, if $b \neq a$, the set A contains exactly one pair $[x, b]$, which was inserted in A when b was marked as "encountered;" if $b = a$, no such pair exists. Additionally, A contains pairs $[b, y]$ which are inserted in A whenever we move from a vertex (b, u) to a vertex (y, u) as we walk along the cycle $\mathcal{H}$, and y has not been marked yet. Since $\mathcal{H}$ visits each vertex of the product P exactly once, exactly $|V_2|$ vertices (b, u) will be visited, $u \in V(G_2)$; each of those may contribute at most one pair $[b, y]$ in A. Hence, the number of pairs in A with one of their elements equal to b is at most $|V_2|+1$, which implies that the degree of the spanning tree T of G_1 is at most $|V_2| + 1$. This contradicts the assumption that $\mathcal{D}(G_1) > |V_2| + 1$ and therefore, P is not hamiltonian. □

Strong results can be obtained if the two factors are traceable (their product is also traceable [10]). The following theorem is due to Behzad and Mahmoodian [2], albeit their proof is quite involved. We provide here a much simpler proof.

Theorem 3 *If both G_1 and G_2 are traceable, then $P = G_1 \times G_2$ is not hamiltonian if and only if both have odd order and none has an odd cycle.*

Proof. If both factors have odd order and none has an odd cycle, by Theorem 1, P is not hamiltonian.

If one of the factors has even order, then P is hamiltonian; a hamiltonian cycle is given in Figure 2(a) (only the spanning paths are considered and shown for each factor). If both have odd order and one has an odd cycle, P is hamiltonian; a hamiltonian cycle is given in Figures 2(b) and 2(c) — it is reproduced from [2]. □

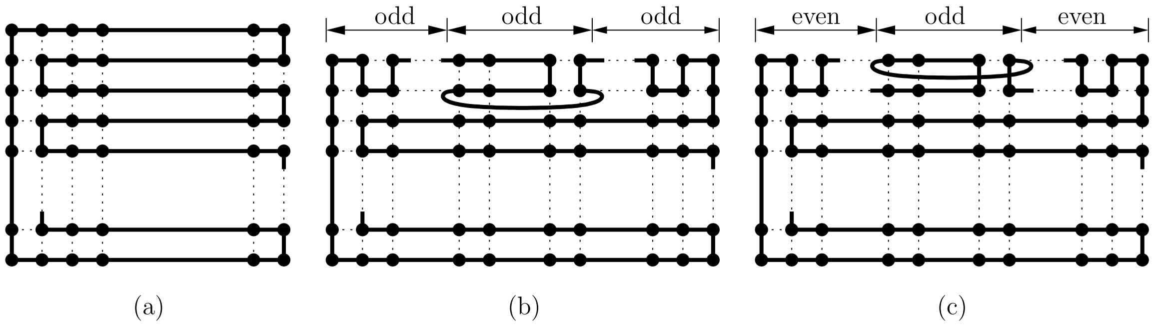


Figure 2: A hamiltonian cycle in the product of two traceable graphs: (a) G_1 has even order; (b) and (c) both factors have odd order and G_2 has an odd cycle (two cases, depending on the position of the cycle).

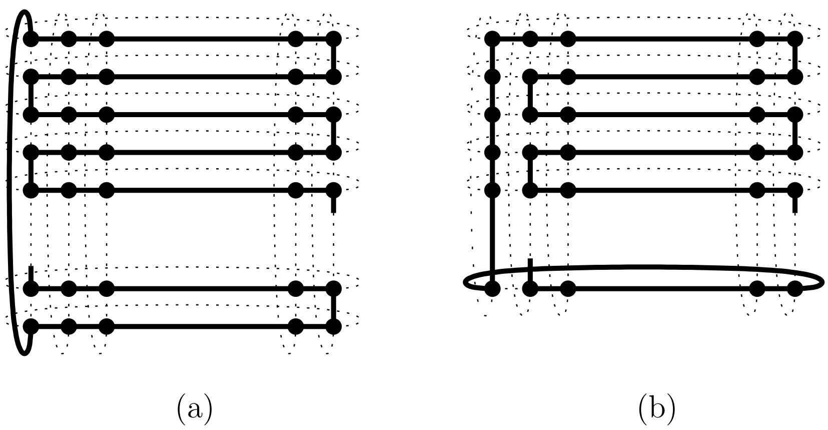


Figure 3: Hamiltonian cycle in the product of two hamiltonian graphs: (a) one having even order and (b) none having even order.

4 Hamiltonian Factors

If both G_1 and G_2 are hamiltonian, it is known that their product is hamiltonian, too. We provide a construction in Figure 3, for the cases of G_1 having (a) an even or (b) an odd number of nodes (only the edges of the hamiltonian cycle in each of the factors are considered and shown).

Consider now the case that only one of the factors is hamiltonian. Assume without loss of generality that G_2 is hamiltonian. To simplify the notation, let $G = G_1$ (as always, assumed connected), $H = G_2$, so that $P = G \times H$ and let

$n = |V(H)|$.

Sabidussi [9] proved that if $|V(G)| \leq (n+2)/2$ then $G \times H$ is hamiltonian. A stronger result was obtained by Rosenfeld and Barnette [8] who showed that if $\Delta(G) \leq n$ then $G \times H$ is hamiltonian. Batagelj and Pisanski [1] showed that this is necessary and sufficient for P to be hamiltonian in the case G is a tree. The following followed as a corollary:

Corollary 1 *If H is hamiltonian and $\mathcal{D}(G) \leq |V(H)|$ then $G \times H$ is hamiltonian.*

In other words, if there exists a spanning tree of G such that its maximum degree is less than n, then $G \times H$ is hamiltonian.

The authors in [1] showed that the converse of Corollary 1 is not true. They defined the *cyclic hamiltonicity*, $\mathrm{cH}(G)$, of a graph G as the minimum n such that $G \times C_n$ is hamiltonian, where C_n is the n-node cycle. Corollary 1 implies $\mathrm{cH}(G) \leq \mathcal{D}(G)$. They finally conjectured that $\mathrm{cH}(G) \leq \mathcal{D}(G) \leq \mathrm{cH}(G) + 1$. What we have proven here, if we combine Corollary 1 with Theorem 2, is that the conjecture is actually true:

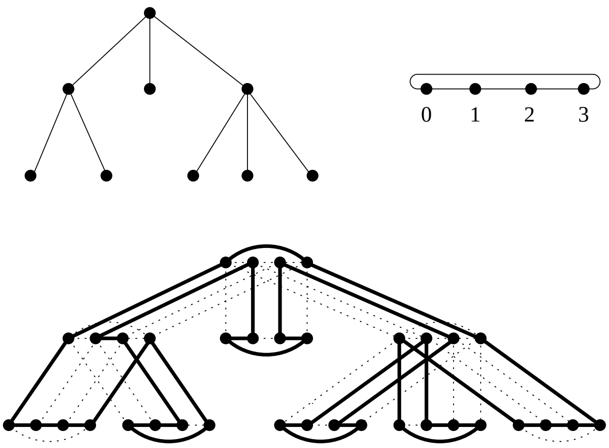
Corollary 2 *For any non-trivial connected graph G,* $\mathrm{cH}(G) \leq \mathcal{D}(G) \leq \mathrm{cH}(G)+1$.

Finally, we provide a linear-time algorithm to construct a hamiltonian cycle in $G \times H$ if the conditions of Corollary 1 are satisfied; T is a given spanning tree of G with $\Delta(T) \leq n$, and $\{i \mid 0 \leq i < n\}$ is the sequence of vertices in a hamiltonian cycle of H. The algorithm is given in Figure 4, where $\oplus$ is addition modulo n, along with an example. A call to trace(u, k, b) adds all vertices $\{(u', v) \mid u'$ belongs to u's subtree in $T\}$ to the partially constructed hamiltonian cycle $\mathcal{H}$. $\mathcal{H}$ enters this set at (u, k) and exits at $(u, k \oplus (-b))$. Since u's degree in T is no more than n, the loop at line 11 will not be executed more than $n-1$ times, therefore i will not advance past the exiting vertex. The loop at line 16 is used to visit all remaining vertices (u, i) and advance to the exit point, in case u's degree in T is less than n.

```
begin main program
    H := ∅
    select any r ∈ V as the root of T
    call trace(r, 0, +1)
end main program

procedure trace(u, k, b)
begin
    append (u, k) to H
    i := k
    for all v: v is a child of u in T do
        call trace(v, i, −b)
        i := i ⊕ b
        append (u, i) to H
    end for
    while i ⊕ b ≠ k do
        i := i ⊕ b
        append (u, i) to H
    end while
end
```

(a)



(b)

Figure 4: (a) An algorithm to construct a hamiltonian cycle in the cartesian product of a tree $T = (V, E)$ and C_n, with $\Delta(T) \leq n$. (b) An example in $T \times C_4$, where the constructed cycle $\mathcal{H}$ is indicated with solid lines.

References

[1] V. Batagelj and T. Pisanski, Hamiltonian cycles in the cartesian product of a tree and a cycle, Discrete Math., 38 (1982) 311–312.

[2] M. Behzad and S. E. Mahmoodian, On topological invariants of the product of graphs, Canad. Math. Bull., 12 (2) (1969) 157–166.

[3] J. - C. Bermond, Hamiltonian graphs, in: Selected topics in graph theory Academic Press, London, (1978) 127–167.

[4] J. - C. Bermond, L. Gargano, A. A. Rescigno and U. Vaccaro, Fast gossiping by short messages, SIAM Journal on Computing, 27 (4) (1998) 917–941.

[5] V. V. Dimakopoulos, On single-port multinode broadcasting, in: IEEE Pacific Rim Conference on Communications, Computers and Signal Processing, Vol. I, Victoria, B.C., Canada, (2001) 75–78.

[6] R. J. Gould, Updating the hamiltonian problem – A survey, Journal of Graph Theory, 15 (2) (1991) 121–157.

[7] F. Harary, Graph Theory, Addison-Wesley, 1969.

[8] M. Rosenfeld and D. Barnette, Hamiltonian circuits in certain prisms, Discrete Math., 5 (1973) 389–394.

[9] G. Sabidussi, Graphs with given group and given graph-theoretical properties, Canad. J. Math., 9 (1957) 515–525.

[10] H. - M. Teichert, On the cartesian sum of undirected graphs, Elektronische Informationsverabeitung und Kybernetik, 18 (12) (1982) 639–646.

[11] J. Zaks, Hamiltonian cycles in products of graphs, Canad. Math. Bull., 17 (5) (1975) 763–765.