On Broadcasting Time*

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Abstract

Broadcasting is an information dissemination problem in which a particular node in a network must transmit an item of information to all the other nodes. In this work we present new lower bounds for the time needed to complete this process in arbitrary graphs. In particular we generalize a result of P. Fraigniaud and E. Lazard [*Discrete Applied Mathematics*, **53** (1994) 79–133] which states that if in a graph there are at least two vertices at distance equal to the diameter from the originator, then broadcasting requires time at least equal to the diameter plus one.

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1 Introduction

Information dissemination, apart from its theoretical interest, has always been of practical importance especially when it comes to network computing and, more recently, distributed-memory multiprocessors. Broadcasting in particular is an essential operation in the majority of parallel algorithms. One important class is parallel numerical algorithms [1, 2].

Consider a connected graph (or *network*) G and a particular vertex (or *node*) v which we will call *source* or *root*. The model we will follow is the *single-port* one whereby each node may utilize only one of its incident links at a time. More formally, broadcasting from v involves the transmission of a message from v to the rest of the vertices in G subject to the following constraints:

- each transmission involves exactly two vertices
- each transmission requires one time unit (or *step*)
- a vertex may call at most one other vertex per time unit
- a vertex may only transmit to an adjacent vertex.

The broadcasting time of a root vertex v, denoted $B(v)$, is the minimum number of time units required to complete broadcasting i.e. to have all other vertices informed about the message. One of the first known results is that since the number of informed vertices can at most double after each step, $B(v) \geq \log_2 n$, where n is the number of vertices in G.

There has been a substantial amount of work on the problem. Hedetniemi, Hedetniemi and Liestman [3] and Fraigniaud and Lazard [4] gave two excellent surveys on the subject. A large portion of the work has been concentrated on constructing efficient broadcast graphs, i.e. graphs that allow for broadcasting in the minimum possible time and contain as few edges as possible; see for example [5] and the references therein. The rest of the work on the problem concentrates mostly on developing minimum-time algorithms or determining bounds on the broadcasting time of particular classes of graphs.

In this paper we derive general lower bounds for the broadcasting problem in arbitrary graphs. Assume that the farthest node from v is at distance $e(v)$ ($e(v)$ is known as the *eccentricity* of node v). Except for $B(v) \geq \log_2 n$, the only other general bound known is that $B(v) \geq e(v)$; this is easy to see since nodes at *any* distance d away from a source node cannot be reached in less than d steps. Fraigniaud and Lazard [4] showed that if there exist at least 2 nodes at distance $e(v)$ from the source node, then at least $e(v)+1$ steps are required. We show here that a general formula may be derived based on the number of nodes at distance greater than a given constant.

2 A New Lower Bound

Consider a graph G and let v be the source node. We will assume that there exist n_i nodes in G which are at distance i from v, for all $i = 1, 2, \ldots, e(v)$. The sequence $\{n_1, n_2, \ldots, n_{e(v)}\}$ is known as the *distance degree sequence* of node v [6]. Based on this sequence, we are going to derive a lower bound for the broadcasting time, $B(v)$, of node v.

Let $r(\ell, t)$ be the *maximum* possible number of distinct nodes that can be reached through paths of length ℓ from v and which get informed exactly at time t. The quantity $r(\ell, t)$ is just an upper bound on the number of such nodes, independently of the the structure of the network. Consider all nodes who got informed some time before t, through a path of length $\ell - 1$. At time t, each of these nodes can at most inform one other node, and this node will receive the message through a path of length ℓ. Therefore $r(\ell, t)$ is described by the following recursion:

$$r(\ell, t) = r(\ell-1, t-1) + r(\ell-1, t-2) + \cdots + r(\ell-1, \ell-1). \qquad (1)$$

Notice that if $\ell > t$ then no node ℓ links away from v has been informed as of time t. Thus $r(\ell, t) = 0$ for $\ell > t$ and this is why the above recursion stops at the term $r(\ell-1, \ell-1)$. The recursion thus holds for any $t \geq \ell$ and $\ell \geq 1$. Since the root can inform at most one of its neighbors at a time, only one node at distance 1 from the root can be informed at any step $t \geq 1$, leading to the boundary condition $r(1, t) = 1$. We finally define $r(0, 0) = 1$; the source is assumed to become aware of the message at time 0. It should be noted that a limited-history version of the above recursion was derived in [7] for graphs with maximum degree of four.

Lemma 1

$$r(\ell, t) = \binom{t-1}{\ell-1}.$$

Proof: Eq. (1) can be written in a more familiar form by observing that $r(\ell-1, t-2) + r(\ell-1, t-3) + \cdots + r(\ell-1, \ell-1) = r(\ell, t-1)$. Consequently we have

$$r(\ell, t) = r(\ell-1, t-1) + r(\ell, t-1),$$

with $r(1, t) = 1$ for $t \geq 1$. Using the standard technique of generating functions, it can easily be shown that the solution of the above recursion is $r(\ell, t) = \binom{t-1}{\ell-1}$ [8]. □

2.1 General Networks

Based on the above, we may express a general lower bound on broadcasting from node v. As we already mentioned, nodes that receive the message through paths of length ℓ are not necessarily at distance ℓ from v. In general, if a vertex is at distance ℓ' from v, it may be informed through any path of length $\ell \geq \ell'$. Concentrate on distance d and assume that broadcasting finishes at time $B(v) = T$. Then by time T all nodes at distance d or more must have been informed. According to the distance degree sequence, the number of nodes at distance at least d from v is

$$R_d = \sum_{i=d}^{e(v)} n_i.$$

Since such nodes must be informed through paths of length at least d, we must have

$$\sum_{\ell=d}^{T} \sum_{t=\ell}^{T} r(\ell, t) \geq R_d.$$

From Lemma 1 we obtain

$$\sum_{\ell=d}^{T} \sum_{t=\ell}^{T} r(\ell, t) = \sum_{\ell=d}^{T} \sum_{t=\ell}^{T} \binom{t-1}{\ell-1}.$$

The sum on the right-hand side evaluates to $\sum_{\ell} \binom{T}{\ell}$ [8]. Hence,

$$\sum_{\ell=d}^{T} \sum_{t=\ell}^{T} r(\ell, t) = \sum_{\ell=d}^{T} \binom{T}{\ell} \geq R_d. \tag{2}$$

We have thus proven the following:

Theorem 2 *If R_d is the number of nodes at distance d or more from v and T is the minimum t such that $\sum_{\ell=d}^{t} \binom{t}{\ell} \geq R_d$ then $B(v) \geq T$, i.e. broadcasting from v requires at least T steps.*

Notice that if $d = 0$, $R_d = n$, i.e. all nodes in the network are included. In this case, using standard results, (2) reduces to

$$\sum_{\ell=0}^{T} \binom{T}{\ell} = 2^T \geq n,$$

which gives $T \geq \log_2 n$, the well-known bound for any network.

At the other extreme, consider the case of $d = e(v)$. If T is equal to $e(v)$ then (2) shows that $R_{e(v)} = n_{e(v)} \leq 1$. This was a result derived in [4]. We may obtain similar results for other values of $T \geq e(v)$. For example, if $T = e(v) + 1$ then (2) gives $\binom{e(v)+1}{e(v)} = e(v) + 1 \geq n_{e(v)}$. Consequently, if there exist more than $e(v) + 1$ nodes at distance $e(v)$ from the source node, broadcasting requires at least $e(v) + 2$ steps.

We next give an approximate formula for the minimum T due to the difficulty associated with handling inequality (2).

Corollary 3 *Broadcasting from v requires time at least equal to*

$$T = \begin{cases} 2d - 1 & \text{if } R_d = 4^{d-1} \\ \log_2(4^{d-1} + R_d) & \text{if } R_d > 4^{d-1} \\ \dfrac{d}{2} + [d!(R_d - 1)]^{1/(d+1)} & \text{if } R_d < 4^{d-1}. \end{cases}$$

Proof: Setting $T = 2d - 1$, (2) evaluates to

$$\sum_{\ell=d}^{2d-1} \binom{2d-1}{\ell} = \frac{1}{2} \sum_{\ell=0}^{2d-1} \binom{2d-1}{\ell} = \frac{1}{2} 2^{2d-1} = 4^{d-1} \geq R_d,$$

hence the first branch of the result.

If $R_d > 4^{d-1}$ then $T > 2d - 1$. We obtain

$$\begin{aligned} \sum_{\ell=d}^{T} \binom{T}{\ell} &= \sum_{\ell=0}^{T} \binom{T}{\ell} - \sum_{\ell=0}^{d-1} \binom{T}{\ell} \\ &\leq \sum_{\ell=0}^{T} \binom{T}{\ell} - \sum_{\ell=0}^{d-1} \binom{2d-1}{\ell} \\ &= 2^T - \frac{1}{2} 2^{2d-1}. \end{aligned}$$

Hence, $2^T \geq 4^{d-1} + R_d$, or $T \geq \log_2(4^{d-1} + R_d)$.

If $R_d < 4^{d-1}$ then $T < 2d-1$. In this case the maximum term of the sum in (2) is $\binom{T}{d}$. Consequently,

$$\sum_{\ell=d}^{T} \binom{T}{\ell} \leq (T-d)\binom{T}{d} + 1.$$

By standard combinatorial properties, $(T-d)\binom{T}{d} = (d+1)\binom{T}{d+1}$. Hence,

$$\sum_{\ell=d}^{T} \binom{T}{\ell} \leq (d+1)\frac{(T-d)(T-d+1)\cdots T}{(d+1)!} + 1.$$

It is known that given m numbers $a_1, \ldots, a_m$, their geometric mean $(a_1 \cdots a_m)^{1/m}$ is less or equal to their arithmetic mean $(a_1 + \cdots + a_m)/m$ [9]. Thus, we seek the minimum T such that

$$\frac{[T - \frac{d}{2}]^{d+1}}{d!} \geq R_d - 1.$$

Taking the $(d+1)$th root of both sides yields the desired result (last branch of the equation).

As a final note, for large m, Stirling's approximation gives $m! \approx \sqrt{2\pi m}(m/e)^m$, where e is the base of the natural logarithms. Consequently, if d is large and $R_d \leq 4^{d-1}$, the last branch of the corollary can be written as

$$T \geq \frac{d}{2} + \frac{d+1}{e}\left(\frac{(R_d-1)\sqrt{2\pi}}{\sqrt{d+1}}\right)^{\frac{1}{d+1}}.$$

□

2.2 Trees

Trees are of particular interest since a broadcasting algorithm actually defines a spanning tree of the underlying network. The interested reader is referred to [10]. If the network is a tree, the path between any two nodes is unique and as a consequence, nodes informed through a path of length ℓ are at distance exactly ℓ from the root; (2) may thus take the simpler form

$$\sum_{t=d}^{T} r(d,t) = \binom{T}{d} \geq n_d,$$

which leads to the following corollary.

Corollary 4 *If the network is a tree rooted at v and T is the minimum t such that $\binom{t}{d} \geq n_d$ then $B(v) \geq T$, i.e. broadcasting from v requires at least T steps.*

It is interesting to note that if $h = e(v)$ is the height of the tree, according to the above corollary, if broadcasting is to be completed in the minimum number of steps (i.e. $T = h$) then there must exist *at most* $\binom{h}{\ell}$ nodes in any level ℓ of the tree. Trees with exactly $\binom{h}{\ell}$ nodes at every level $\ell = 0, 1, \ldots, h$ and which achieve this minimum broadcasting time are unique and are known as *binomial trees* [10].

3 Summary

In conclusion, we derived lower bounds on the time needed to broadcast from a vertex in arbitrary networks. The bounds may be viewed as a generalization of a result in [4] and they are based on the distance degree sequence of the source node. The known bounds from the literature, $B(v) \geq \log_2 n$ and $B(v) \geq e(v)$ become special cases of our formulas.

Tighter bounds may be derived if it is known that the network is of bounded degree, i.e. no node has degree greater than a given constant. In this case one may consider a recursion similar to (1) but with limited history. However to the best of our knowledge no closed-form solution can be obtained for such a recursion. In [7] an approximation was given for the case where the maximum degree is four. The only other known result for bounded-degree graphs is due to J.-C. Bermond *et al* [5] and is based only on the number of nodes in the network.

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